* TFNM

Def'n A TFNM is a time series regression that predicts current values of a dependent variable based on both the current values as well as the lagged values of one or more explanatory variables.

· Form:

$$y_t = \alpha + V_0 x_t + V_1 x_{t-1} + V_2 x_{t-2} + \cdots + \varepsilon_t$$
$$= \alpha + \sum_{i=0}^{\infty} V_i x_{t-i} + \varepsilon_t \leftarrow \text{called "the system."}$$

IRC

DLM

Def'n The coefficients V_t's are referred to as the impulse response function of the system.

Note Distributed lag models require all impulse response functions of the same sign.

The system is stable $\iff \sum_{i=0}^{\infty} |V_i| < \infty$.

$g := \sum_{i=0}^{\infty} V_i$ is called the steady-state gain.

(represents the impact on Y when X_{t-i}'s are held constn over time)

· Modelling:

→ Unstructured

LASSO

→ Structured estimation / approximation

e.g. Almon distributed lag models (finite DLM)

$$V_j = \sum_{j=0}^{n} a_j i^j, \quad i = 0, \dots, k, \quad n < k.$$

→ e.g. Koyck DLM (infinite)

$$V_i = \beta\lambda^i \ (|\lambda| < 1 \text{ perhaps}), \quad \varepsilon_t = \sum_{i=0}^{\infty} \lambda^i \xi_{t-i}.$$

So the system is

$$y_t = \alpha + \sum_{i=0}^{\infty} \beta\lambda^i x_{t-i} + \sum_{i=0}^{\infty} \lambda^i \xi_{t-i}.$$

We can approximate the Koyck distributed lag model using the following ARX model:

$$y_t = \alpha + \lambda y_{t-1} + \beta x_t + \xi_t .$$

So, Koyck DLM: ($|\lambda|<1$)

$$\begin{aligned} y_t &= \alpha + \sum_{i=0}^{\infty} \beta \lambda^i x_{t-i} + \sum_{i=0}^{\infty} \lambda^i \xi_{t-i} \\ &= \alpha + \sum_{i=0}^{\infty} \beta (\lambda B)^i x_t + \sum_{i=0}^{\infty} (\lambda B)^i \xi_t \\ &= \alpha + \frac{\beta}{1-\lambda B} x_t + \frac{\xi_t}{1-\lambda B} . \end{aligned}$$

→ Rational DLM

$$y_t = \sum_{i=0}^{\infty} v_i x_{t-i} + \varepsilon_t = v(B) x_t + \varepsilon_t .$$

$$v(B) = \sum_{i=0}^{\infty} v_i B^i .$$

$v(B)$ can be approximated by a ratio of two polynomials:

$$v(B) = \frac{\delta_0 + \delta_1 B + \cdots + \delta_r B^r}{1 - \theta_1 B - \cdots - \theta_s B^s} = \frac{\delta_r(B)}{\theta_s(B)} .$$

Approximate y_t by this: $y_t = \frac{\delta_r(B)}{\theta_s(B)} x_t + \varepsilon_t$.

We allow $\varepsilon_t \sim$ ARMA (stationary).

· Model building process

→ Building a single input TFNM includes:

① Preliminary identification of IRC's: v_i's.

② Specification of the noise term ε_t.

③ 〃 TF using a rational polynomial in B if necessary.

④ Estimation of the TFN model specified in ② and ③.

⑤ Model diagnostic checks.

Step ① . Preliminary identification (prewhitening)

The system: $y_t = v(B)x_t + \varepsilon_t$. (or $\dot{y}_t := y_t - a$)

Suppose $x_t \sim ARMA$: $\Phi(B)x_t = \Theta(B)\alpha_t$, $\alpha_t \overset{iid}{\sim} N(0, \sigma_\alpha^2)$

Multiply $\frac{\Phi(B)}{\Theta(B)}$ on each side of the system:

$$\tau_t := \frac{\Phi(B)}{\Theta(B)} y_t = V(B) \cdot \underbrace{\frac{\Phi(B)}{\Theta(B)} x_t}_{= \alpha_t} + \underbrace{\frac{\Phi(B)}{\Theta(B)} \varepsilon_t}_{=: n_t} .$$

$$\Rightarrow \quad \tau_t = V(B)\alpha_t + n_t , \quad \alpha_t \perp\!\!\!\perp n_t .$$

Again, multiply each side by α_{t-j} $(j \geq 0)$:

$$\tau_t \alpha_{t-j} = V(B)\alpha_t \alpha_{t-j} + n_t \alpha_{t-j}$$

Take $E(\cdot)$:

$$Cov(\tau_t, \alpha_{t-j}) = V_j \cdot Var(\alpha_{t-j})$$

$$\Rightarrow V_j = \frac{Cov(\tau_t, \alpha_{t-j})}{Var(\alpha_{t-j})} = Corr(\tau_t, \alpha_{t-j}) \cdot \frac{se(\tau_t)}{se(\alpha_t)} .$$

We can examine the significance of V_j by examining the sig. of $Corr(\tau_t, \alpha_{t-j})$.

Step ⑤ . Diagnostic check

Consider TFNM: $y_t = \frac{\omega(B)}{\delta(B)} B^b x_t + N_t$, $N_t \sim ARMA$.

So $N_t \sim ARMA(p, q) \Rightarrow \Phi_p(B)N_t = \Theta_q(B)a_t$, $a_t \overset{iid}{\sim} N(0, \sigma_a^2)$.

From TFNM, $y_t = \frac{\omega(B)}{\delta(B)} x_{t-b} + N_t$

$$\Rightarrow \hat{N}_t = y_t - \frac{\hat{\omega}(B)}{\hat{\delta}(B)} x_{t-b} .$$

$=:$ ~~$y_t - \hat{v}(B)x_{t-b}$~~ $y_t - \hat{v}(B)x_t$.

That is, $\hat{v}(B) = \frac{\hat{\omega}(B)}{\hat{\delta}(B)} B^b$.

So $\Phi_p(B)\hat{N}_t = \Theta_q(B)\hat{a}_t$

$$\Rightarrow \frac{\Phi_p(B)}{\Theta_q(B)}\hat{N}_t = \hat{a}_t, \quad \begin{cases}\Phi_p(B) = 1-\phi_1B-\cdots-\phi_pB^p \\ \Theta_q(B) = 1+\theta_1B+\cdots+\theta_qB^q.\end{cases}$$

Two checks

① CCR check: checking whether the noise $\{a_t\}$ and the input series $\{x_t\}$ are uncorrelated.

$$Q_0 = m(m+2)\sum_{j=0}^{K}(m-j)^{-1}\hat{\rho}^2_{\hat{\alpha}\hat{a}}(j) \sim \chi^2_{K+1-M}.$$

$\alpha_t = \frac{\Phi(B)}{\Theta(B)}x_t$ is obtained from prewhitening.

$\hat{a}_t = \frac{\Phi(B)}{\Theta_q(B)}\hat{N}_t$, $m = n-t_0+1$. So m is the # of res. $\hat{a}_t$ calculated.

~~M = # of parameters estimated in the noise model.~~

M = # of param.s δ_i and w_i estimated in the transfer function $v(B) = \frac{w(B)}{\delta(B)}$.

② ACR check: checking whether the noise model is adequate.

$$Q_1 = m(m+2)\sum_{j=1}^{K}(m-j)^{-1}\hat{\rho}^2_{a}(j) \sim \chi^2_{K-(p+q)}$$

Step ④ · Box-Tiao transformation

$$y_t = V(B)x_t + n_t, \quad n_t \sim ARMA(n,n).$$

So $\Phi_n(B) n_t = \Theta_n(B) a_t$, $a_t \overset{iid}{\sim} N(0, \sigma_a^2)$.

$$\Rightarrow \frac{\Phi_n(B)}{\Theta_n(B)} n_t = a_t.$$

$\Rightarrow$ (Multiply $\frac{\Phi_n}{\Theta_n}$ on both sides)

$$\frac{\Phi_n(B)}{\Theta_n(B)} y_t =: \mathcal{Y}_t = V(B) \frac{\Phi_n(B)}{\Theta_n(B)} x_t + \frac{\Phi_n(B)}{\Theta_n(B)} n_t$$

$$=: V(B)\mathcal{X}_t + a_t.$$

① Run OLS reg. on $y_t = \sum_{j=b}^{s} V_j x_{t-j} + e_t$ and collect residuals $\{\hat{e}_t\}$, [illegible]

② Identify an ARMA model for $\{\hat{e}_t\}$.

③ Apply Box-Tiao transformation to x_t and y_t.

④ Run OLS reg on $\mathcal{Y}_t = \sum_{j=b}^{s} V_j \mathcal{X}_{t-j} + a_t$.

⑤ Check whether the reg. res. on ④ are serially correlated. If not, repeat ② to ④.

2018 Winter Lin Midterm practice

Introduction

1. Define weak stationarity of a time series.

→ A time series $\{X_t\}_T$ is weakly stationary if:

① $E(|X_t|^2) < \infty$, ② $E(X_t) = m \ \forall t \in T$,

③ $\gamma_x(r,s) = \gamma_x(r+t, s+t) \ \forall r,s,t \in T$.

2. Define classical decomposition.

→ It is a statistical task that deconstructs $\{X_t\}$ into several components, each representing one of the underlying categories of patterns. Components include trend, seasonal variation, cyclical changes, and irregular fluctuations. Trend is a long-term change in the mean level; seasonal variation is a variation that occurs at specific regular intervals (annually, quarterly, or daily); cyclical changes reflect repeated but not periodic fluctuations due to some other physical cause. Irregular components, or "noises", at time t describes random, irregular changes. They represent the residuals or remainder of $\{x_t\}$ after the other components have been removed.

3. State the stops of modelling time series data.

→ ① Plot the time series and check for trend, seasonal and cyclical variations. Also, check for apparent sharp changes in behaviour, and outliers.

② Remove trend and seasonal / cyclical componenes to get residuals.

③ Choose a model to fit the residuals.

④ Conduct forecasting (the residuals), and invert the transformation carried out in ②.

4. Consider $x_t = \frac{1}{3}a_t + \frac{1}{3}a_{t-1}$, $a_t \overset{iid}{\sim} N(0, \sigma_a^2)$, and $y_t = \frac{1}{2}x_t - \frac{1}{2}x_{t-2}$.

a) Calculate ACVF for x_t and y_t.

So $Cov(x_t, x_{t+k}) = Cov(\frac{1}{3}a_t + \frac{1}{3}a_{t-1}, \frac{1}{3}a_{t+k} + \frac{1}{3}a_{t+k-1})$.

$k=0 \Rightarrow Var(x_t) = \frac{1}{9}\sigma_a^2 + \frac{1}{9}\sigma_a^2 = \underline{\underline{\frac{2}{9}\sigma_a^2}}$

$k=1 \Rightarrow Cov(\frac{1}{3}a_t + \frac{1}{3}a_{t-1}, \frac{1}{3}a_{t+1} + \frac{1}{3}a_t)$

$= \underline{\underline{\frac{1}{9}\sigma_a^2}}$

$k=2 \Rightarrow Cov(\frac{1}{3}a_t + \frac{1}{3}a_{t-1}, \frac{1}{3}a_{t+2} + \frac{1}{3}a_{t+1}) = \underline{\underline{0}}$

$k>2 \Rightarrow Cov(x_t, x_{t+k}) = 0$.

So for $\{x_t\}$, $\gamma_0 = \frac{2\sigma_a^2}{9}$, $\gamma_1 = \frac{\sigma_a^2}{9}$, $\gamma_k = 0 \ \forall k \geq 2$.

This makes sense $\because 3x_t \sim MA(1)$.

$+\frac{1}{4}\cdot\frac{1}{2}\cdot-\frac{1}{2}$

$y_t = \frac{1}{2}x_t - \frac{1}{2}x_{t-2}$,

$Cov(y_t, y_t) = Var(y_t) = Var(\frac{1}{2}x_t - \frac{1}{2}x_{t-2})$

$= \frac{1}{4}Var(x_t) + \frac{1}{4}Var(x_{t-2}) - \frac{1}{2}Cov(x_t, x_{t-2})$

$= \frac{1}{4}\gamma_0 + \frac{1}{4}\gamma_0 - \frac{1}{2}\gamma_2$

$= \frac{1}{2}\gamma_0 = \underline{\frac{\sigma_a^2}{9}}$. //

$Cov(y_t, y_{t+1}) = Cov(\frac{1}{2}x_t - \frac{1}{2}x_{t-2}, \frac{1}{2}x_{t+1} - \frac{1}{2}x_{t-1})$

$= \frac{1}{4}\gamma_1 - \frac{1}{4}\gamma_3 \cancel{- \frac{1}{4}\gamma_1 + \frac{1}{4}\gamma_1} = \frac{1}{4}\gamma_1$

$= \underline{\frac{\sigma_a^2}{36}}$, //

$Cov(y_t, y_{t+2}) = Cov(\frac{1}{2}x_t - \frac{1}{2}x_{t-2}, \frac{1}{2}x_{t+2} - \frac{1}{2}x_t)$

$= \frac{1}{4}[\gamma_2 - \gamma_4 - \gamma_0 + \gamma_2]$

$= -\frac{\gamma_0}{4} = -\frac{1}{\cancel{4}2}\cdot\frac{2\sigma_a^2}{9} = \underline{-\frac{\sigma_a^2}{18}}$ //

$Cov(y_t, y_{t+3}) = Cov(\frac{1}{2}x_t - \frac{1}{2}x_{t-2}, \frac{1}{2}x_{t+3} - \frac{1}{2}x_{t+1})$

$= \frac{1}{4}[\gamma_3 - \gamma_5 - \gamma_1 + \gamma_3]$

$= \frac{1}{4}\times -\gamma_1 =$ [illegible] $\underline{-\frac{\sigma_a^2}{36}}$ //

$Cov(y_t, y_{t+k}) = \underline{0}$ // $\forall k \geq 4$.

b) Calculate $\rho_{xy}(k)$'s.

$\rho_{xy}(0,0) = \frac{Cov(x_t, y_t)}{\sqrt{Var(x_t)Var(y_t)}}$. $Var(x_t) = \gamma_0 = \frac{2\sigma_a^2}{9}$.

$Var(y_t) = \frac{\gamma_0}{2}$

$Cov(x_t, y_t) = Cov(x_t, \frac{1}{2}x_t - \frac{1}{2}x_{t-2})$

$= \frac{1}{2}Var(x_t) - \frac{1}{2}\gamma_2 = \frac{1}{2}\gamma_0$

So $\rho_{xy}(0,0) = \frac{\frac{\gamma_0}{2}}{\frac{\gamma_0}{\sqrt{2}}} = \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}} = \underline{\frac{\sqrt{2}}{2}}$ //

$\gamma_X(t_1, t_2) = E[(X_{t_1} - \mu_{X_{t_1}})(X_{t_2} - \mu_{X_{t_2}})]$

~~$\rho_{XY}(0,1) = \frac{Cov(X_t, y_{t+1})}{\sqrt{Var(X_t) Cov(y_t, y_{t+1})}}$~~

$\rho_{XY}(0,1) = \frac{\gamma_{XY}(0,1)}{\sqrt{\gamma_X(0,0)\gamma_Y(1,1)}}$.

Since y_t is stationary ($\because$ sum of stationary processes),
$\gamma_Y(1,1) = \gamma_Y(1-1, 1-1) = \gamma_Y(0) = Var(y_t)$.

So $\rho_{XY}(0,1) = \frac{\gamma_{XY}(0,1)}{\frac{\gamma_0}{\sqrt{2}}}$.

$Cov(X_t, y_{t+1}) = Cov(X_t, \frac{1}{2}X_{t+1} - \frac{1}{2}X_{t-1})$
$= \frac{1}{2}\gamma_1 - \frac{1}{2}\gamma_1 = 0$.
$\therefore \underline{\rho_{XY}(0,1) = 0.}_{11}$

$\rho_{XY}(1,0) = \frac{Cov(X_{t+1}, y_t)}{\frac{\gamma_0}{\sqrt{2}}}$, $Cov(X_{t+1}, y_t) = Cov(X_{t+1}, \frac{1}{2}X_t - \frac{1}{2}X_{t-2})$

$= \frac{1}{2}\gamma_1 - \frac{1}{2}\gamma_3 = \frac{1}{2}\gamma_1$.
$\Rightarrow \rho_{XY}(1,0) = \frac{\frac{\gamma_1}{2}}{\frac{\gamma_0}{\sqrt{2}}} = \frac{\sqrt{2}\gamma_1}{2\gamma_0} = \frac{\sqrt{2}}{2} \cdot \frac{\frac{\cancel{\sigma^2}}{9}}{\frac{2\sigma^2}{9}} = \underline{\frac{\sqrt{2}}{4}.}_{11}$

$\rho_{XY}(1,1)$

$\gamma_{XY}(k) = Cov(X_t, y_{t+k})$
$= Cov(X_t, \frac{1}{2}X_{t+k} - \frac{1}{2}X_{t+k-2})$
$= \frac{1}{2}[\gamma_k - \gamma_{k-2}]$.

$\rho_{XY}(k) =$ ~~$\frac{\gamma_{XY}(k)}{\sqrt{\gamma_X(k)\gamma_Y(k)}}$~~ $\frac{\gamma_{XY}(k)}{\sqrt{\gamma_X(k,k)\gamma_Y(k,k)}} = \frac{\gamma_{XY}(k)}{\sqrt{\gamma_X(0)\gamma_Y(0)}}$.
$= \frac{\frac{1}{2}[\gamma_k - \gamma_{k-2}]}{\frac{\gamma_0}{\sqrt{2}}}$.

$k=0 \Rightarrow \rho_{XY}(0) = \frac{\frac{\gamma_0}{2}}{\frac{\gamma_0}{\sqrt{2}}} = \underline{\frac{\sqrt{2}}{2}}_{11}$, $\underline{\rho_{XY}(1) = 0.}_{11}$

$\rho_{XY}(2) = \frac{-\frac{\gamma_0}{2}}{\frac{\gamma_0}{\sqrt{2}}} = \underline{-\frac{\sqrt{2}}{2}}_{11}$, $\rho_{XY}(3) = \frac{-\frac{\gamma_1}{2}}{\frac{\gamma_0}{\sqrt{2}}} = \underline{-\frac{\sqrt{2}}{4}}_{11}$

$\rho_{XY}(k) = \underline{0}_{11}$ if $k \geq 4$.

$$\gamma_{xy}(r,s) = \mathrm{Cov}(x_{t+r}, y_{t+s})$$
$$= \mathrm{Cov}(x_{t+r}, \tfrac{1}{2}x_{t+s} - \tfrac{1}{2}x_{t+s-2})$$
$$= \tfrac{1}{2}[\gamma_{s-r} - \gamma_{s-r-2}].$$

$$\rho_{xy}(r,s) = \frac{\frac{1}{2}[\gamma_{s-r} - \gamma_{s-r-2}]}{\frac{\gamma_0}{\sqrt{2}}} = \frac{\sqrt{2}[\gamma_{s-r} - \gamma_{s-r-2}]}{2\gamma_0}$$

$$\gamma_{xy}(0,k) = \mathrm{Cov}(x_t, y_{t+k}).$$

$$\rho_{xy}(0,k) = \frac{\gamma_{xy}(0,k)}{\frac{\gamma_0}{\sqrt{2}}} = \frac{\frac{1}{2}\gamma_k - \frac{1}{2}\gamma_{k-2}}{\frac{\gamma_0}{\sqrt{2}}} = \begin{cases} \sqrt{2}/2 & k=0 \\ 0 & k=1 \\ -\sqrt{2}/2 & k=2 \\ -\sqrt{2}/4 & k=3 \\ 0 & k \geq 4 \end{cases}$$

$$\gamma_{xy}(k,0) = \mathrm{Cov}(x_{t+k}, y_t)$$
$$= \mathrm{Cov}(x_{t+k}, \tfrac{1}{2}x_t - \tfrac{1}{2}x_{t-2})$$
$$= \tfrac{1}{2}[\gamma_k - \gamma_{k+2}]$$

$$\rho_{xy}(k,0) = \frac{\frac{1}{2}\gamma_k - \frac{1}{2}\gamma_{k+2}}{\frac{\gamma_0}{\sqrt{2}}} = \begin{cases} \sqrt{2}/2 & k=0 \\ \sqrt{2}/4 & k=1 \\ 0 & k=2 \\ 0 & \vdots \\ 0 & \end{cases} \qquad \frac{\frac{\gamma_1}{2}}{\frac{\gamma_0}{\sqrt{2}}} = \frac{\sqrt{2}\gamma_1}{2\gamma_0} = \frac{\sqrt{2}}{2}\cdot\frac{}{\gamma_0}$$

$\rho_{xy}(t_1,t_2)$ $t_1 \leftarrow r$ $s \to t_2$	…	-3	-2	-1	0	1	2	3	4	5	…
⋮	…										
-3		$\frac{\sqrt{2}}{2}$									
-2			$\frac{\sqrt{2}}{2}$								
-1				$\frac{\sqrt{2}}{2}$							
0					$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{4}$	0	0	0	0	…
1					0	$\frac{\sqrt{2}}{2}$					
2					$-\sqrt{2}/2$		$\frac{\sqrt{2}}{2}$				
3					$-\sqrt{2}/4$			$\frac{\sqrt{2}}{2}$			
4					0				$\frac{\sqrt{2}}{2}$		
5					0					$\frac{\sqrt{2}}{2}$	
⋮					⋮						

$$\therefore \rho_{xy}(r,s) = \frac{\frac{1}{2}[\gamma_{s-r} - \gamma_{s-r-2}]}{\frac{\gamma_0}{\sqrt{2}}}$$
$$= \frac{\sqrt{2}[\gamma_{s-r} - \gamma_{s-r-2}]}{2\gamma_0}.$$

⟶ //

5. $\{Z_t\}_{t=1,\dots,n}$ is a zero-mean weakly stationary time series w/ $E(Z_t)=0$, $Var(Z_t)=\sigma^2$, $\gamma_k=E(Z_tZ_{t+k})$, and $\rho_k=\gamma_k/\gamma_0$.

a) Show that $Var(\bar{Z})=\frac{\gamma_0}{n}\sum_{k=-(n-1)}^{n-1}\left(1-\frac{|k|}{n}\right)\rho_k$, $\bar{Z}=\frac{1}{n}\sum_{t=1}^{n}Z_t$.

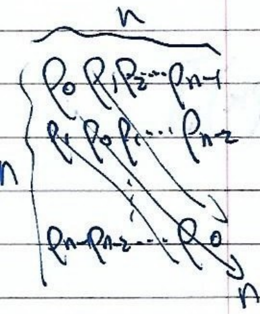
$$Var(\bar{Z})=Var\left(\frac{1}{n}\sum_{t=1}^{n}Z_t\right)=\frac{1}{n^2}Var\left(\sum_{t=1}^{n}Z_t\right)$$

$$=\frac{1}{n^2}\sum_{r=1}^{n}\sum_{s=1}^{n}Cov(Z_r,Z_s)=\frac{1}{n^2}\sum_{r=1}^{n}\sum_{s=1}^{n}\gamma_{|r-s|}$$

$$=\frac{\gamma_0}{n^2}\sum_{r=1}^{n}\sum_{s=1}^{n}\rho_{|r-s|}$$

$$=\frac{\gamma_0}{n^2}\left[\sum_{r=1}^{n}\left(\rho_{|r-1|}+\rho_{|r-2|}+\cdots+\rho_{|r-n|}\right)\right]$$

$$=\frac{\gamma_0}{n^2}\left[\rho_{|1-1|}+\rho_{|1-2|}+\cdots+\rho_{|1-n|}+\rho_{|2-1|}+\rho_{|2-2|}+\cdots+\rho_{|n-n|}\right]$$



$$=\frac{\gamma_0}{n^2}\left[\rho_0+\rho_1+\cdots+\rho_{n-1}+\rho_1+\rho_0+\rho_1+\cdots+\rho_{n-2}+\cdots+\rho_0\right]$$

$$=\frac{\gamma_0}{n^2}\left[n\rho_0+2(n-1)\rho_1+2(n-2)\rho_2+\cdots+2(n-(n-1))\rho_{n-1}\right]$$

$$=\frac{\gamma_0}{n^2}\left[n\rho_0+\sum_{k=1}^{n-1}(n-|k|)\rho_{|k|}+\sum_{k=1}^{n-1}(n-|k|)\rho_{|k|}\right]$$

$$=\frac{\gamma_0}{n^2}\left[n\rho_0+\sum_{k=-(n-1)}^{-1}(n-|k|)\rho_{|k|}+\sum_{k=1}^{n-1}(n-|k|)\rho_{|k|}\right]$$

$$=\frac{\gamma_0}{n}\left[\rho_0+\sum_{k=-(n-1)}^{-1}\left(1-\frac{|k|}{n}\right)\rho_{|k|}+\sum_{k=1}^{n-1}\left(1-\frac{|k|}{n}\right)\rho_{|k|}\right]$$

$$=\frac{\gamma_0}{n}\left[\sum_{k=-(n-1)}^{n-1}\left(1-\frac{|k|}{n}\right)\rho_{|k|}\right],$$

//

b) $E(\hat{S}^2) = \frac{n}{n-1}\gamma_0 - \frac{n}{n-1}Var(\bar{Z})$. Then,

$$\frac{E(\hat{S}^2)}{\gamma_0} = 1 - \frac{2}{n-1}\left[\sum_{k=1}^{n-1}\rho_k - \frac{1}{n}\sum_{k=1}^{n-1}k\rho_k\right].$$ Why?

$$Var(\bar{Z}) = \frac{\gamma_0}{n}\left[\sum_{k=-(n-1)}^{n-1}\left(1-\frac{|k|}{n}\right)\rho_{|k|}\right].$$

So $\frac{Var(\bar{Z})}{\gamma_0} = \frac{1}{n}\sum_{k=-(n-1)}^{n-1}\left(1-\frac{|k|}{n}\right)\rho_{|k|}$,

thus $\frac{n}{n-1}\cdot\frac{Var(\bar{Z})}{\gamma_0} = \frac{1}{n-1}\sum_{k=-(n-1)}^{n-1}\left(1-\frac{|k|}{n}\right)\rho_{|k|}$

Thus $\frac{E(\hat{S}^2)}{\gamma_0} = \frac{n}{n-1} - \frac{1}{n-1}\sum$

$$= \frac{1}{n-1}\left[n - \sum_{k=-(n-1)}^{n-1}\left(1-\frac{|k|}{n}\right)\rho_{|k|}\right] \Big\} \times$$

$$= \frac{n}{n-1}\left[1 - \sum_{k=-(n-1)}^{n-1}(n-|k|)\rho_{|k|}\right]$$

$$= \frac{n}{n-1}\left[1 - \left(n\underset{=1}{\rho_0} + \sum_{k=1}^{n-1}(n-k)\rho_k + \sum_{k=1}^{n-1}(n-k)\rho_k\right)\right]$$

$$= \frac{n}{n-1}\left[1 - \left(n + 2\sum_{k=1}^{n-1}(n-k)\rho_k\right)\right]$$

$$= \frac{n}{n-1}\left[1 - n - 2\left(\sum_{k=1}^{n-1}n\rho_k - \sum_{k=1}^{n-1}k\rho_k\right)\right]$$

$$= \frac{n}{n-1}\left[1 - n - \frac{2n}{\text{[illegible]}}\sum_{k=1}^{n-1}\rho_k - 2\sum k\rho_k\right]$$

$$= \frac{n}{n-1} - \frac{n^2}{n-1} - \frac{2}{n-1}\sum\rho_k - \frac{2n}{n-1}\sum k\rho_k$$

$$= \frac{n(1-n)}{n-1} - \frac{2}{n-1}\left[\sum\rho_k - \right.$$

$$\frac{E(\hat{S}^2)}{\gamma_0} = \frac{n}{n-1} - \frac{n}{n-1}\frac{Var(\bar{Z})}{\gamma_0}$$

$$= \frac{n}{n-1}\left[1 - \frac{Var(\bar{Z})}{\gamma_0}\right]$$

$$= \frac{n}{n-1}\left[1 - \frac{1}{n}\sum_{k=-(n-1)}^{n-1}\left(1-\frac{|k|}{n}\right)\rho_{|k|}\right]$$

$$= \frac{n}{n-1}\left[1 - \frac{1}{n}\left[2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k + \rho_0\right]\right]$$

$$= \frac{n}{n-1}\left[1 - \frac{2}{n}\left[\sum_{k=1}^{n-1}\rho_k - \sum_{k=1}^{n-1}\frac{k\rho_k}{n}\right] - \frac{\rho_0}{n}\right]$$

$$= \frac{n}{n-1}\left[1 - \frac{2}{n}\sum\rho_k + \frac{2}{n}\cdot\frac{1}{n}\sum k\rho_k - \frac{1}{n}\right]$$

$$= \frac{n}{n-1}\left[\frac{n-1}{n} - \frac{2}{n}\left[\sum\rho_k - \frac{1}{n}\sum k\rho_k\right]\right]$$

$$= \underline{1 - \frac{2}{n-1}\left[\sum\rho_k - \frac{1}{n}\sum k\rho_k\right]}, \checkmark, //$$

ARMA models

1. State the Wold decomposition. Discuss the reason Wold decomposition supports the use of ARMA models for modelling time series.

→ It is the theorem that says any zero-mean process $\{X_t\}$ which isn't deterministic can be expressed as a sum of $X_t = U_t + V_t$, where, $\{U_t\}$ denotes an MA(∞) process and ~~∞~~ $\{V_t\}$ is a deterministic process uncorrelated w/ $\{U_t\}$. Processes are called deterministic if the values X_{n+j}, $j \geq 1$, of the process $\{X_t \mid t \in \mathbb{Z}\}$ are perfectly predictable in terms of $\mu_n \in \text{span}\{X_t \mid -\infty < t \leq n\}$. The theorem supports a time series process $\{X_t\}$ to be approximated by a linear model which is what ARMA model does w/ a white noise $\{a_t\}$.

2. Consider $X_t \sim ARMA(2,2)$.

$\Phi_2(B)X_t = \Theta_2(B)a_t$, $a_t \overset{iid}{\sim} N(0,\sigma^2)$, $\sigma^2 = 1$.

$$\Phi_2(B)x_t = \Theta_2(B)a_t.$$

a) Under which condition is $\{x_t\}$ causal/stationary?

$\rightarrow$ $\{x_t\}$ is stationary iff

① $\Phi_2(B)$'s roots all lie outside of the unit circle, iff

② $x_t = \Psi_\infty(B)a_t$, $\Psi_\infty(B) = \frac{\Theta_2(B)}{\Phi_2(B)}$, and $\Psi_\infty(B)$ has coefficients $\sum|\psi_j| < \infty$.

b) Under which condition is $\{x_t\}$ invertible? Why do we care?

$\rightarrow$ $\{x_t\}$ is invertible iff

① $\Theta_2(B)$'s roots all lie outside of the unit circle, iff

② $\Pi_\infty(B)x_t = a_t$, $\Pi_\infty(B) = \frac{\Phi_2(B)}{\Theta_2(B)}$, and $\Pi_\infty(B)$ has coefficients $\sum|\pi_j| < \infty$.

$\rightarrow$ We consider an invertible ARMA model because the invertible process implies $|\theta_i| < 1$ and $|\theta_i|$ converging to 0, i.e. the most recent observations have higher weight ~~than~~ observations from the more distant past. This makes sense and is intuitively correct.

c) Suppose x_t is causal, i.e. $x_t = \Psi_\infty(B)a_t$. $x_t = a_t + \sum_{j=1}^{\infty}\psi_j a_{t-j}$. Calculate ψ_j's, $j=1,\cdots,6$.

~~→~~

$\rightarrow$ $\Phi_2(B) = 1-\phi_1B-\phi_2B^2$, $\Theta_2(B) = 1+\theta_1B+\theta_2B^2$.

$$\Psi_\infty(B) = \frac{\Theta_2(B)}{\Phi_2(B)} = \frac{1+\theta_1B+\theta_2B^2}{1-\phi_1B-\phi_2B^2}.$$

$$1+\alpha_1B+\gamma_1B^2$$

$$1-\phi_1B-\phi_2B^2 \overline{\big)\,1+\theta_1B+\theta_2B^2}$$

$$\underline{1-\phi_1B-\phi_2B^2}$$

$$\alpha_1B+\alpha_2B^2, \quad \alpha_1=\theta_1+\phi_1, \ \alpha_2=\theta_2+\phi_2.$$

$$\underline{\alpha_1B-\beta_1B^2-\beta_2B^3} \quad \beta_1=\alpha_1\phi_1, \ \beta_2=\alpha_1\phi_2.$$

$$\gamma_1B^2+\beta_2B^3 \quad \gamma_1=$$

$$1-\phi_1 B-\phi_2 B^2 \,\big)\, \overline{1+\theta_1 B+\theta_2 B^2} \quad \text{quotient: } 1+\psi_1 B+\psi_2 B^2+\psi_2 B^3$$

$$\underline{1-\phi_1 B-\phi_2 B^2}$$

$$\psi_1 B+\alpha_1 B^2 \qquad \underline{\psi_1=\theta_1+\phi_1},\ \alpha_1=\theta_2+\phi_2.$$

$$\underline{\psi_1 B-\beta_1 B^2-\beta_2 B^3} \quad \beta_1=\phi_1\psi_1,\ \beta_2=\phi_2\psi_1.$$

$$\psi_2 B^2+\beta_2 B^3 \qquad \psi_2=\alpha_1+\beta_1=\underline{\alpha_1+\phi_1\psi_1}=\underline{\theta_2+\phi_2+\phi_1\psi_1}$$

$$\underline{\psi_2 B^2-\gamma_1 B^3-\gamma_2 B^4} \quad \gamma_1=\phi_1\psi_2,\ \gamma_2=\phi_2\psi_2$$

$$\psi_3 B^3+\gamma_2 B^4 \qquad \psi_3=\beta_2+\gamma_1$$

$$\underline{\psi_3 B^3-\delta_1 B^4-\delta_2 B^5} \qquad =\beta_2+\phi_1\psi_2$$

$$\psi_4 B^4+\delta_2 B^5 \qquad =\underline{\phi_2\psi_1+\phi_1\psi_2}$$

$$\delta_1=\phi_1\psi_3,\ \delta_2=\phi_2\psi_3.$$

$$\psi_4=\gamma_2+\delta_1$$

$$=\underline{\phi_2\psi_2+\phi_1\psi_3.}$$

Inductively,
$$\begin{cases}\psi_1=\theta_1+\phi_1, \\ \psi_2=\theta_2+\phi_2+\phi_1\psi_1 \\ \psi_k=\phi_2\psi_{k-2}+\phi_1\psi_{k-1},\ k\geq 3. \ /\!/\end{cases}$$

d) Suppose x_t is invertible, i.e. $\pi_\infty(B)x_t=a_t$.
$x_t-\sum_{j=1}^{\infty}\pi_j x_{t-j}=a_t$. Calculate π_j, $j=1,\cdots,6$.

$$\longrightarrow \pi_\infty(B)=\frac{\Phi_2(B)}{\Theta_2(B)}.$$

$$1+\theta_1 B+\theta_2 B^2 \,\big)\, \overline{1-\phi_1 B-\phi_2 B^2} \quad \text{quotient: } 1+\pi_1 B+\pi_2 B^2+\pi_3 B^3$$

$$\underline{1+\theta_1 B+\theta_1 B^2}$$

$$\pi_1 B+\alpha_1 B^2 \qquad \pi_1=-\phi_1-\theta_1,\ \alpha_1=-\phi_2-\theta_1$$

$$\underline{\pi_1 B+\beta_1 B^2+\beta_2 B^3} \qquad \beta_1=\pi_1\theta_1,\ \beta_2=\pi_1\theta_2$$

$$\pi_2 B^2-\beta_2 B^3 \qquad \pi_2=\alpha_1-\beta_1=-\phi_2-\theta_1-\pi_1\theta_1$$

$$\underline{\pi_2 B^2+\gamma_1 B^3+\gamma_2 B^4} \qquad \gamma_1=\pi_2\theta_1,\ \gamma_2=\pi_2\theta_2$$

$$\pi_3 B^3-\gamma_2 B^4 \qquad \pi_3=-\beta_2-\gamma_1$$

$$\underline{\pi_3 B^3+\delta_1 B^4+\delta_2 B^5} \qquad =\underline{-\pi_1\theta_2-\pi_2\theta_1}$$

$$\pi_4 B^4-\delta_2 B^5 \qquad \delta_1=\pi_3\theta_1,\ \delta_2=\pi_3\theta_2.$$

$$\pi_4=-\gamma_2-\delta_1$$

$$=\underline{-\pi_2\theta_2-\pi_3\theta_1}$$

Inductively, $\pi_1=-\phi_1-\theta_1$, $\pi_2=-\phi_2-\theta_1-\pi_1\theta_1$,
$\underline{\pi_k=-\pi_{k-2}\theta_2-\pi_{k-1}\theta_1,\ k\geq 3.}$ //

3. Consider $x_t = \theta_0 a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2}$, $a_t \overset{iid}{\sim} N(0,1)$.

a) Calculate ACF of x_t for lags 1, 2, and 3.

$$\rightarrow \gamma_x(0) = Var(x_t)$$
$$= Var(\theta_0 a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2})$$
$$= \theta_0^2 \times 1 + \theta_1^2 \times 1 + \theta_2^2 \times 1$$
$$= \underline{\theta_0^2 + \theta_1^2 + \theta_2^2}.$$

$$\gamma_x(1) = Cov(x_t, x_{t+1})$$
$$= Cov(\theta_0 a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2}, \theta_0 a_{t+1} + \theta_1 a_t + \theta_2 a_{t-1})$$
$$= \theta_0\theta_1 \times 1 + \theta_1\theta_2 \times 1$$
$$= \underline{\theta_0\theta_1 + \theta_1\theta_2},$$

$$\gamma_x(2) = Cov(\theta_0 a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2}, \theta_0 a_{t+2} + \theta_1 a_{t+1} + \theta_2 a_t)$$
$$= \underline{\theta_0\theta_2}$$

$$\gamma_x(k) = \underline{0} \quad \forall k \geq 3$$

So $\underline{\rho_x(0) = 1,\ \rho_x(1) = \frac{\gamma_x(1)}{\gamma_x(0)},\ \rho_x(2) = \frac{\gamma_x(2)}{\gamma_x(0)},}$
$\underline{\rho_x(k) = 0 \quad \forall k \geq 3.}$

b) Calculate ϕ_{kk}, $k = 1, 2, 3$.

$$\rightarrow \therefore \phi_{11} = \underline{\rho_1 = \gamma_x(1)/\gamma_x(0).}$$

~~$\begin{bmatrix} \rho_0 \\ \vdots \\ \rho_n \end{bmatrix} = \begin{bmatrix} \rho_0 \cdots \rho_n \\ \rho_0\rho_1 \cdots \rho_{n-1} \\ \vdots \\ \rho_n \cdots \rho_0 \end{bmatrix} \begin{bmatrix} \phi_{n1} \\ \phi_{n2} \\ \vdots \\ \phi_{nn} \end{bmatrix}$~~

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_n \end{bmatrix} = \begin{bmatrix} \rho_0 & \rho_1 & \cdots & \rho_{n-1} \\ \rho_1 & \rho_0\rho_1 & \cdots & \rho_{n-2} \\ & \vdots & & \\ \rho_{n-1} & \cdots & & \rho_0 \end{bmatrix} \begin{bmatrix} \phi_{n1} \\ \phi_{n2} \\ \vdots \\ \phi_{nn} \end{bmatrix}$$

$$\phi_{22} = \frac{\det\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\det\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}} = \underline{\frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}},$$

$$\phi_{33} = \frac{\det\begin{vmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{vmatrix}}{\det\begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}} = \frac{1 \cdot \begin{vmatrix} 1 & \rho_2 \\ \rho_1 & \rho_3 \end{vmatrix} - \rho_1 \cdot \begin{vmatrix} \rho_1 & \rho_2 \\ \rho_2 & \rho_3 \end{vmatrix} + \rho_1 \cdot \begin{vmatrix} \rho_1 & 1 \\ \rho_2 & \rho_1 \end{vmatrix}}{1 \cdot \begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix} - \rho_1 \cdot \begin{vmatrix} \rho_1 & \rho_1 \\ \rho_2 & 1 \end{vmatrix} + \rho_2 \cdot \begin{vmatrix} \rho_1 & 1 \\ \rho_2 & \rho_1 \end{vmatrix}}$$

$$= \underline{\frac{\rho_3 - \rho_1\rho_2 - \rho_1(\rho_1\rho_3 - \rho_2^2) + \rho_1(\rho_1^2 - \rho_2)}{1 - \rho_1^2 - \rho_1(\rho_1 - \rho_1\rho_2) + \rho_2(\rho_1^2 - \rho_2)}}.$$

So $\phi_{33} = \dfrac{\rho_3 - \rho_1\rho_2 - \rho_1^2\rho_3 + \rho_1\rho_2^2 + \rho_1^3 - \rho_1\rho_2}{1 - \rho_1^2 - \rho_1^2 + \rho_1^2\rho_2 + \rho_1^2\rho_2 - \rho_2^2}$

$= \dfrac{\rho_3 - 2\rho_1\rho_2 - \rho_1^2\rho_3 + \rho_1\rho_2^2 + \rho_1^3}{1 - 2\rho_1^2 + 2\rho_1^2\rho_2 - \rho_2^2}$ //

4. Consider $x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} = a_t \overset{iid}{\sim} N(0,1)$.

a) Calculate ACFs, $k=1,2,3$.

Multiply both by x_t:

$x_t^2 - \phi_1 x_t x_{t-1} - \phi_2 x_t x_{t-2} = a_t x_t$.

Take $E(\cdot)$: $\gamma_0 - \phi_1\gamma_1 - \phi_2\gamma_2 = \sigma_a^2 = 1$.

$E(a_t x_t) = E(a_t(\phi_1 x_{t-1} + \phi_2 x_{t-2} + a_t))$

$= E(\phi_1 a_t x_{t-1} + \phi_2 a_t x_{t-2} + a_t^2)$

$= 0 + 0 + \sigma_a^2 = 1$.

So $\gamma_0 - \phi_1\gamma_1 - \phi_2\gamma_2 = 1$.

Now, if x_{t-k} is multiplied $(k \neq 0)$,

$x_t x_{t-k} - \phi_1 x_{t-1}x_{t-k} - \phi_2 x_{t-2}x_{t-k} = a_t x_{t-k}$

and taken $E(\cdot)$:

$\gamma_k - \phi_1\gamma_{k-1} - \phi_2\gamma_{k-2} = 0$, $k \geq 1$.

$k=1 \Rightarrow \gamma_1 - \phi_1\gamma_0 - \phi_2\gamma_1 = 0$

$k=2 \Rightarrow \gamma_2 - \phi_1\gamma_1 - \phi_2\gamma_0 = 0$

So $\begin{bmatrix} 1 & -\phi_1 & -\phi_2 \\ -\phi_1 & 1-\phi_2 & 0 \\ -\phi_2 & -\phi_1 & 1 \end{bmatrix}\begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$\dfrac{1+\phi_2-1}{1+\phi_2} = \dfrac{\phi_2}{1+\phi_2}$

$$\left[\begin{array}{ccc|c} 1 & -\phi_1 & -\phi_2 & 1 \\ -\phi_1 & 1-\phi_2 & 0 & 0 \\ -\phi_2 & -\phi_1 & 1 & 0 \end{array}\right] \xrightarrow[-R_1+R_3]{} \left[\begin{array}{ccc|c} 1 & -\phi_1 & -\phi_2 & 1 \\ -\phi_1 & 1-\phi_2 & 0 & 0 \\ -(1+\phi_2) & 0 & 1+\phi_2 & -1 \end{array}\right]$$

$$\xrightarrow[\frac{1}{1+\phi_2}R_3]{\frac{1}{-\phi_1}R_2} \left[\begin{array}{ccc|c} 1 & -\phi_1 & -\phi_2 & 1 \\ 1 & \frac{\phi_2-1}{\phi_1} & 0 & 0 \\ -1 & 0 & 1 & \frac{-1}{1+\phi_2} \end{array}\right] \xrightarrow[\substack{R_3+R_2 \\ -R_3}]{R_3+R_1} \left[\begin{array}{ccc|c} 0 & -\phi_1 & -\phi_2+1 & 1-\frac{1}{1+\phi_2} \\ 0 & \frac{\phi_2-1}{\phi_1} & 1 & \frac{-1}{1+\phi_2} \\ 1 & 0 & -1 & \frac{1}{1+\phi_2} \end{array}\right]$$

$$\xrightarrow[-\frac{1}{\phi_1}R_1]{\frac{\phi_1}{\phi_2-1}R_2} \left[\begin{array}{ccc|c} 0 & 1 & \frac{\phi_2-1}{\phi_1} & \frac{-\phi_2}{+\phi_1(1+\phi_2)} \\ 0 & 1 & \frac{\phi_1}{\phi_2-1} & \frac{-\phi_1}{(1+\phi_2)(\phi_2-1)} \\ 1 & 0 & -1 & \frac{1}{1+\phi_2} \end{array}\right]$$

$$\frac{\phi_1}{\underset{(\phi_2+1)(\phi_2-1)}{\phi_2^2-1}} - \frac{\phi_2}{\phi_1(1+\phi_2)}$$

$$= \frac{\phi_1^2 - \phi_2(\phi_2-1)}{(\phi_2+1)(\phi_2-1)\phi_1}$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & \frac{1}{1+\phi_2} \\ 0 & 1 & \frac{\phi_2-1}{\phi_1} & \frac{-\phi_2}{\phi_1(1+\phi_2)} \\ 0 & 1 & \frac{\phi_1}{\phi_2-1} & \frac{-\phi_1}{(\phi_2-1)(\phi_2+1)} \end{array}\right]$$

$$\xrightarrow{-R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & \frac{1}{1+\phi_2} \\ 0 & 0 & \frac{-\phi_1}{\phi_2-1}+\frac{\phi_2-1}{\phi_1} & \frac{\phi_1^2-\phi_2^2+\phi_2}{(\phi_2^2-1)\phi_1} \\ 0 & 1 & \frac{\phi_1}{\phi_2-1} & \frac{-\phi_1}{(\phi_2-1)(\phi_2+1)} \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & \frac{1}{1+\phi_2} \\ 0 & 1 & \frac{\phi_1}{\phi_2-1} & \frac{-\phi_1}{\phi_2^2-1} \\ 0 & 0 & \frac{-\phi_1^2+(\phi_2-1)^2}{\phi_1(\phi_2-1)} & \frac{\phi_1^2-\phi_2^2+\phi_2}{\phi_1(\phi_2-1)(\phi_2+1)} \end{array}\right]$$

$\phi_1^2 - \phi_2(\phi_2-1)$

$$\xrightarrow{\frac{\phi_1(\phi_2-1)}{-\phi_1^2+(\phi_2-1)^2}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & \frac{1}{1+\phi_2} \\ 0 & 1 & \frac{\phi_1}{\phi_2-1} & \frac{-\phi_1}{(\phi_2-1)(\phi_2+1)} \\ 0 & 0 & 1 & \frac{\phi_1^2-\phi_2^2+\phi_2}{((\phi_2-1)^2-\phi_1^2)(\phi_2+1)} \end{array}\right]$$

$$\frac{\cancel{\phi_1(\phi_2-1)}}{-\phi_1^2+(\phi_2-1)^2} \times \frac{\phi_1^2-\phi_2^2+\phi_2}{\cancel{\phi_1(\phi_2-1)}(\phi_2+1)} = \frac{\phi_1^2-\phi_2^2+\phi_2}{(\phi_2+1)(-\phi_1^2+(\phi_2-1)^2)}$$

$$\xrightarrow[-\frac{\phi_1}{\phi_2-1}R_3+R_2]{R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{\phi_1^2-\phi_2^2+\phi_2}{(\phi_2+1)((\phi_2-1)^2-\phi_1^2)} + \frac{1}{1+\phi_2} \\ 0 & 1 & 0 & \frac{-\phi_1(\phi_1^2-\phi_2^2+\phi_2)}{(\phi_2^2-1)((\phi_2-1)^2-\phi_1^2)} - \frac{\phi_1}{(\phi_2-1)(\phi_2+1)} \\ 0 & 0 & 1 & \frac{\phi_1^2-\phi_2^2+\phi_2}{(\phi_2+1)((\phi_2-1)^2-\phi_1^2)} \end{array}\right].$$

Or, $\gamma_0 - \phi_1\gamma_1 - \phi_2\gamma_2 = 1$

$\Rightarrow 1 - \phi_1\rho_1 - \phi_2\rho_2 = 1, \quad \underline{(\rho_0 = 1).}$ //

$\gamma_1 - \phi_1\gamma_0 - \phi_2\gamma_1 = 0$

$\Rightarrow \rho_1 - \phi_1 - \phi_2\rho_1 = 0$

$\Rightarrow (1-\phi_2)\rho_1 = \phi_1 \Rightarrow \underline{\rho_1 = \frac{\phi_1}{1-\phi_2}}$ //

$\gamma_2 - \phi_1\gamma_1 - \phi_2\gamma_0 = 0$

$\Rightarrow \rho_2 - \phi_1\rho_1 - \phi_2 = 0$

$\Rightarrow \rho_2 = \phi_1\rho_1 + \phi_2 = \underline{\frac{\phi_1^2}{1-\phi_2} + \phi_2}.$ //

$\underline{(k \geq 1) \quad \rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}, \ \rho_0 = 1.}$ //

b) Calculate PACF's, $k = 1, 2, 3.$

$\phi_{11} = \rho_1 = \underline{\frac{\phi_1}{1-\phi_2}}.$ //

$\phi_{22} = \underline{\frac{\rho_2 - \rho_1^2}{1-\rho_1^2}}.$ // $\quad \underline{\phi_{kk} = 0 \ \forall k \geq 3}$ //

Aside:

$$\gamma_0 = \frac{\cancel{\phi_1^2} \cancel{- \phi_2^2} \cancel{+\phi_2}}{(\phi_2+1)((\phi_2-1)^2 - \phi_1^2)} + \frac{\cancel{\phi_2^2} - 2\phi_2 + 1 \cancel{-\phi_1^2}}{(\phi_2+1)((\phi_2-1)^2-\phi_1^2)}$$

$$= \underline{\frac{1-\phi_2}{(\phi_2+1)((\phi_2-1)^2-\phi_1^2)}}. \ //$$

$-\phi_1[\phi_2^2 - 2\phi_2 + 1 - \phi_1^2]$

$$\gamma_1 = \frac{-\phi_1^3 + \phi_1\phi_2^2 - \phi_1\phi_2}{(\phi_2^2-1)((\phi_2-1)^2-\phi_1^2)} - \cancel{\frac{\phi_1(\phi_2+1)}{(\phi_2^2-1)(\phi_2+1)}} \ \frac{\phi_1}{(\phi_2^2-1)}$$

$$= \frac{-\phi_1^3 + \phi_1\phi_2^2 - \phi_1\phi_2 - \phi_1[(\phi_2-1)^2-\phi_1^2]}{(\phi_2^2-1)[(\phi_2-1)^2-\phi_1^2]}$$

$$= \frac{\cancel{-\phi_1^3} \cancel{+\phi_1\phi_2^2} - \phi_1\phi_2 \cancel{-\phi_1\phi_2^2} + 2\phi_1\phi_2 - \phi_1 \cancel{+\phi_1^3}}{(\phi_2^2-1)[(\phi_2-1)^2-\phi_1^2]}$$

$$= \frac{\phi_1\phi_2 - \phi_1}{\sim\sim} = \frac{\phi_1\cancel{(\phi_2-1)}}{\cancel{(\phi_2-1)}(\phi_2+1)[\sim\sim]}$$

$$= \underline{\frac{\phi_1}{(\phi_2+1)[(\phi_2-1)^2-\phi_1^2]}}. \ // \qquad \underline{\gamma_2 = \frac{\phi_1^2 - \phi_2^2 + \phi_2}{(\phi_2+1)[(\phi_2-1)^2-\phi_1^2]}}. \ //$$

5. Define two portmanteau tests for checking model adequacy.

→ The overall tests that check an entire group of residuals ACFs, assuming the model is adequate, are called portmanteau tests. They test whether any group of ACFs of the res. are diff. from 0. Two tests are: Box and Pierce w/ $Q_{BP} = n\sum_{k=1}^{m}\hat{\rho}_k^2 \dot\sim \chi^2_{m-(p+q)}$, and Ljung and Box, $Q_{LB} = n(n+2)\sum_{k=1}^{m}(n-k)^{-1}\hat{\rho}_k^2 \sim \chi^2_{m-(p+q)}$.

$\hat{\rho}_k$ = ACF of res. at lag k.

$\hat{\rho}_k = \frac{\sum_{t=1}^{n-k}\hat{a}_t\hat{a}_{t+k}}{\sum_{t=1}^{n}\hat{a}_t^2}$

$y_t = \alpha + \beta x_t + e_t$

$\{x_t\}$ and $\{y_t\}$ are stationary + indep. →

(Based on the assumptions that $\rho_{XY}(k) = 0 \ \forall k$
$\Rightarrow \hat{\rho}_{XY}(k) \sim N(0, \frac{1}{n}(1+2\sum_{j=-\infty}^{\infty}\rho_X(j)\rho_Y(j)))$, and
~~$\sqrt{n}\,\hat{\rho}_{XY}(k) \dot\sim N(0, 1+2\sum_{j=0}^{\infty}\rho_X(j)\rho_Y(j))$~~, and
$\hat{\rho}_X(k) \dot\sim N(0, \frac{1}{n}), k \neq 0$).

If ⊕ no serial corr. $\Rightarrow \hat{\gamma}_{XY}(k) \dot\sim N(0, \frac{1}{n})$.
size of sample

~~Tests~~ These tests are practical, and req. are minimal to use the fitted model, but it lacks power if comparing w/ traditional statistical tests, such as LRT. To overcome, we can do finite sample adjustments, complicated functional of res. ACFs, Monte Carlo test, or portmanteau tests for randomness and ARMA models w/ infinite variance innovations.

6. Define AIC and BIC criteria for model selection. Discuss which of them tends to select a more parsimonious model.

→ AIC and BIC are estimators of the relative quality of statistical models for a given set of data. AIC is defined as $AIC := -2\log ML + 2k$, and $BIC := -2\log ML + k\log(n)$, where $\log ML$ is the maximized value of log-likelihood function for a model fitted to a given dataset, and k is the # of (independently adjusted) parameters w/i the model. BIC puts more penalties on the # of parameters used by fitted models, and some empirical studies indicate that the model selected by BIC performs better in the post-sample analysis, such as forecasting. The model w/ the lowest BIC is preferred.

Box-Jenkins approach and Unit root tests.

1. Suppose $x_t = \alpha + \beta t^3 + y_t$, $y_t = a_t + \theta_1 a_{t-1}$, $a_t \overset{iid}{\sim} N(0,1)$. Show that $(1-B)^3 x_t$ is stationary.

$\rightarrow$ $(1-B)^3 = (1-2B+B^2)(1-B) = (1-2B+B^2-B+2B^2-B^3)$

$= 1-3B+3B^2-B^3$

$(1-3B+3B^2-B^3)x_t$

$= (1-3B+3B^2-B^3)(\alpha+\beta t^3+y_t)$

$\therefore (1-B)^3\alpha = \alpha-3\alpha+3\alpha-\alpha = 0.$

$(1-B)^3\beta t^3 = \beta t^3 - 3\beta(t-1)^3 + 3\beta(t-2)^3 - \beta(t-3)^3$

$= \beta t^3 - 3\beta(t^3-3t^2+3t-1) + 3\beta(t^3-6t^2+12t-8) - \beta(t-3)^3$

$= \beta t^3 + 9\beta t^2 - 9\beta t + 3\beta - 18\beta t^2 + 36\beta t - 24\beta - \beta(t-3)^3$

$= \beta t^3 - 9\beta t^2 + 27\beta t - 21\beta - \beta(t^3-9t^2+27t-27)$

$= -21\beta + 27\beta = \underline{6\beta}.$ //

$(t^2-4t+4)(t-2)$

$t^3-4t^2+4t-2t^2+8t-8$

$t^3-6t^2+12t-8$

$(t^2-6t+9)(t-3)$

$t^3-6t^2+9t-3t^2+18t-27$

$\underline{(1-B)^3 y_t \text{ is stationary } \because y_t \text{ is stationary}}$ //

2. Define ARIMA(p,d,q). Use the def'n of ARIMA to explain the idea of the unit root test and discuss the def'n of I(d) process.

$\rightarrow$ $\{x_t\}$ is said to follow an ARIMA model of order (p,d,q) if $W_t = (1-B)^d x_t$ follows an ARMA(p,q) model. Mathematically, we have $\Phi_p(B)(1-B)^d x_t = \Theta_q(B) a_t$, $a_t \overset{iid}{\sim} N(0, \sigma_a^2)$.

The unit root test tests whether a time series variable is non-stationary and possesses a unit root. We say a process has a unit root if 1 is a root of the characteristic equation. For example, with $\Phi_p(B) x_t = a_t$ (AR(p)), $\Phi_p(B) = 1 - \sum_{i=1}^{p} \phi_i B^i$, if 1 is a root of $\Phi_p(x)$, then we say x_t is integrated of order one, or I(1) process. If 1 is a root w/ multiplicity d, we say x_t is a I(d) process. Put in another way, x_t following a stationary ARMA model after differencing d times is said to be integrated of order d, or I(d) process.

3. Define the basic Dickey-Fuller test. What are issues related to the basic Dickey-Fuller test taught in class?

→ The (basic) DF test tests H_0 whether the ~~a~~ unit root is present in an AR model. ~~[illegible]~~ H_1 is usually stationarity or trend stationarity. Here's the procedure: consider $x_t \sim AR(1)$, i.e.

$$x_t - \phi_1 x_{t-1} = a_t \Rightarrow x_t = \phi_1 x_{t-1} + a_t, \quad a_t \overset{iid}{\sim} N(0, \sigma_a^2).$$

$$\text{So } x_t - x_{t-1} = \phi_1 x_{t-1} + a_t - x_{t-1}$$
$$= (\phi_1 - 1) x_{t-1} + a_t$$
$$\Rightarrow \nabla x_t = \pi x_{t-1} + a_t, \quad \pi = \phi_1 - 1.$$

$$H_0: \pi = 0 \quad \text{vs} \quad H_1: \pi > 0 \quad (I(1) \text{ vs. } I(0)).$$

a= reg. intercept

$\tau^T = (\alpha_1, \alpha_2, \cdots)$

$DR_t = (t, t^2, t^3, \cdots)$

We reject H_0 ~~of~~ for small p-value or large t-obs about $\hat{\pi}$. In general, $\nabla x_t = a + \tau^T DR_t + \pi x_{t-1} + a_t$.

→ There are some issues regarding DF test: the DF test considers only a single unit root, and it assumes a correct model specification, i.e. correct specification of time trend and intercept. The DGP may have contained both AR and MA terms, and

... there might've been structural breaks in the data, all of which this test doesn't consider.

4. Define the augmented DF test (ADF). Discuss how to select the length of lag for the ADF test.

→ The ADF test tests H_0 that a unit root is present in a time series after carrying out an estimation on time trends and using the lagged ∇X_{t-j} terms to correct ~~the~~ effect of autocorrelated error terms. Here's the procedure:
$\nabla X_t = \tau^T DR_t + \pi X_{t-1} + \sum_{j=1}^{k} r_j \cdot \nabla X_{t-j} + a_t$ is the encompassing ADF test equation. $K = p-1$.
$H_0: \pi = 0$ vs. $H_1: \pi \neq 0$. We reject H_0 for small p-value or large t-statistic about $\hat{\pi}$.

→ There are two ways to select the lag length for the ADF test. One is an autoregression approximation: an unknown ARIMA$(p,1,q)$ process can be often approximated by an ARIMA$(K,1,0)$ process with $k \leq T^{1/3}$, where T is the series length of time series. The other is a general-to-specific methodology: you start w/ a relatively long lag length (say p^*) and conduct the usual t- or F-test. If the statistic of lag p^* is insignificant at some specified critical value, re-estimate the regression using the length p^*-1. Repeat the process until the last is significantly different from zero. Once our tentative lag length is determined, conduct the diagnostic check by plotting residual autocorrelation plot and portmanteau tests on regression residuals to ensure that the choice of the lag length is correct.

5. State how to test the presence of two unit roots in a time series.

→ Instead of $\nabla x_t = \pi_1 x_{t-1} + a_t$, write $\nabla^2 x_t = \pi_1 \nabla x_{t-1} + a_t$. Then use the appropriate statistic to determine whether π_1 is significantly different from 0. If $\pi_1 \approx 0$, i.e. H_0 not rejected, then conclude $\{x_t\} \sim I(2)$. ($H_0: \pi_1 = 0$ vs $H_1: \pi_1 \neq 0$). If $\pi_1 \not\approx 0$, go on determine whethere there is a single unit root: $\nabla^2 x_t = \pi_1 \nabla x_{t-1} + \pi_2 x_{t-1} + a_t$. $H_0: \pi_1 < 0$ and $\pi_2 = 0$ (presence of single unit root). Here we can use the DF critical values to test the [illegible] null hypothesis $\pi_2 = 0$. If we reject the null hypothesis, we can conclude that $\{x_t\}$ is stationary.

Transfer function noise model

Consider an infinite distributed lag model:

(1) $y_t = \sum_{i=0}^{K} v_i x_{t-i} + a_t$, $a_t \overset{iid}{\sim} N(0,1)$, $t = 1, \cdots, T$,

(2) $(1-\phi B) x_t = e_t$, $e_t \overset{iid}{\sim} N(0,1)$.

1. Determine K in (1) using the prewhitening procedure.

→ Assume x_t is invertible. ~~Write $x_t = (\frac{1}{1-\phi B} e_t))$ with $\pi(B) = 1-\phi B$~~ Write $\left(\frac{1-\phi B}{1}\right) x_t = e_t$
$= \pi(B) x_t = e_t$.

Multiply $\pi(B)$ on both sides of (1). Then,

$\pi(B) y_t = V(B)\pi(B) x_t + \pi(B) a_t$

$= \tau_t = V(B) e_t + \varepsilon_t$, $V(B) = v_0 + v_1 B + \cdots + v_K B^K$.

Now multiply e_{t-j}, $j = 0, 1, 2, \cdots$. We then have:

$\tau_t e_{t-j} = V(B) e_t e_{t-j} + \varepsilon_t e_{t-j}$.

Take $E(\cdot)$ on both sides to obtain:

$Cov(\tau_t, e_{t-j}) = V_j Var(e_{t-j})$ (we are assuming $\{e_t\}$ is indep. (uncorrelated) of [illegible] ε_s $\forall t, s$.). (equivalently, $\{x_t\} \perp\!\!\!\perp a_s$ $\forall t, s$ or $\{e_t\} \perp\!\!\!\perp a_s$).

That is, $V_j = \frac{Cov(\tau_t, e_{t-j})}{Var(e_{t-j})} = \rho_{\tau e}(j) \times \frac{se(\tau_t)}{se(e_t)}$.

We can examine the statistical significance of V_j by examining the significance of $\rho_{\tau e}(j)$ using the standard (sample) cross-correlation functions. We can use this result to identify K.

<u>Aside</u>$^{\#1}$ How do we test the "assumption" in 1?

<u>Ans.</u>$^{\#1}$ ① Look at the standard (sample) cross-correlation plot between $\{e_t\}$ and $\{a_t\}$, i.e. ρ_{ea}, OR,

or $e_t \perp\!\!\!\perp \varepsilon_s$ $\forall t, s$.

② Look at the following portmanteau test:

$Q_0 = m(m+2) \sum_{j=0}^{k^*} (m-j)^{-1} \hat{\rho}^2_{\hat{e}\hat{\varepsilon}}(j) \sim \chi^2_{k^*+1-M}$, where

m = # of residuals ($\hat{\varepsilon}_t$) calculated $= T$, $\hat{e}$ obtained from the prewhitening process, M = # of parameters estimated in the transfer function $= k$, $k^* < T$.

k^* = # of lags being tested.

<u>Aside</u>$^{\#2}$ How do we know if ε_t follows $N(0,1)$?

<u>Ans</u>$^{\#2}$ Look at:

$$Q_1 = m(m+2) \sum_{j=1}^{k^*} (m-j)^{-1} \hat{\rho}^2_{\varepsilon}(j) \sim \chi^2_{k^*-(p+q)}.$$

p and q are from $a_t \sim ARMA(p,q)$.

Since $a_t \overset{iid}{\sim} N(0,1)$, let $p=q=0$.

2. Suppose $k=\infty$ and $V_i = V^i$, $i = 0, 1, 2, \cdots$.

a) Approximate Equation (1) using the rational distributed lag model. Specifically, let

$$y_t = \frac{\delta(B)}{\vartheta(B)} x_t + \xi_t, \begin{cases} \delta(B) = \delta_0 + \delta_1 B + \cdots + \delta_r B^r \\ \vartheta(B) = 1 - \vartheta_1 B - \cdots - \vartheta_s B^s \end{cases}.$$

Specify $\delta(B)$, $\vartheta(B)$, ξ_t in terms of V_i, a_t, and B.

→ $K = \infty$, and $V_i = V^i$, $i = 0, 1, \cdots$. So,

$y_t = \sum_{i=0}^{\infty} V_i x_{t-i} + a_t$ ($a_t \overset{iid}{\sim} N(0,1)$) ($t = 1, \cdots, T$)

$= V_\infty(B) x_t + a_t$, and $x_t \sim ARMA(1,0)$,

$(1 - \phi B) x_t = e_t$, $e_t \overset{iid}{\sim} N(0,1)$.

$V_\infty(B) = \sum_{i=0}^{\infty} V_i B^i = \sum_{i=0}^{\infty} (VB)^i = \frac{1}{1-VB}$, given $|V| < 1$.

So $y_t = \frac{1}{1-VB} x_t + a_t$, $a_t = \sum_{i=0}^{\infty} V^i \xi_{t-i}$ (assumption)

$= \sum_{i=0}^{\infty} (VB)^i \xi_t$

$= \frac{\xi_t}{1-VB}$

Thus $y_t = \sum_{i=0}^{\infty} V^i x_{t-i} + \sum_{i=0}^{\infty} V^i \xi_{t-i}$

$= \sum_{i=0}^{\infty} (VB)^i x_t + \sum_{i=0}^{\infty} (VB)^i \xi_t$

$= \frac{x_t}{1-VB} + \frac{\xi_t}{1-VB}$

$\underline{\delta(B) = 1,\ \vartheta(B) = 1 - VB,\ \xi_t = a_t.}$ //

b) Suppose $\delta(B) = 1$, $\vartheta(B) = 1 - \vartheta B$. State how to estimate ϑ using the Box and Tiao transformation.

→ $\xi_t = (1 - VB) a_t \Rightarrow (1 - VB) a_t = \xi_t$, $\xi_t \overset{iid}{\sim} N(0,1)$ ← assumption.

$y_t = V_\infty(B) x_t + a_t$

$\Rightarrow (1 - VB) y_t = V_\infty(B)(1 - VB) x_t + (1 - VB) a_t$

$\Rightarrow \tilde{y}_t = V_\infty(B) \tilde{x}_t + \xi_t$. $V(B) \approx \frac{1}{1 - \vartheta B}$.

① Run OLS regression on $y_t = \frac{1}{1-VB} x_t + a_t$ and collect $\{\hat{a}_t\}$.

② Fit an ARMA(p,q) to $\hat{a}_t$.

③ Apply Box-Tiao as above.

④ Run OLS reg. on $\tilde{y}_t = \frac{1}{1-VB} \tilde{x}_t + \xi_t$.

⑤ Check whether the reg. res in ④ are serially correlated. If not, repeat ② to ④.

Let $\underline{\hat{\vartheta} = \vartheta.}$ //

C) State how to use portmanteau tests to check model adequacy.

→ Two important features in $y_t = V(B)x_t + a_t$, $(1-\phi B)x_t = e_t$, $a_t \overset{iid}{\sim} N(0,1)$, $e_t \overset{iid}{\sim} N(0,1)$, are:

① whether x_t and a_t are uncorrelated, and

② whether a_t is indeed a white noise.

From $\tau_t = (1-\phi B)y_t = V(B)e_t + \varepsilon_t$, we can conduce the following tests:

① $Q_0 = m(m+2)\sum_{j=0}^{K^*}(m-j)^{-1}\hat{\rho}^2_{\hat{e}\hat{\varepsilon}}(j) \sim \chi^2_{K^*+1-M}$,
where m = # of residuals $(\hat{\varepsilon}_t)$ calculated,
K^* = # of testing lags, M = # of parameters ~~identified in prewhitening procedure~~, estimated in transfer function noise model, and $\hat{e}_t$'s obtained in prewhitening procedure.

$\varepsilon_t = (1-\phi B)a_t$

② $Q_1 = m(m+2)\sum_{j=0}^{K^*}(m-j)^{-1}\hat{\rho}^{\varepsilon}(j) \sim \chi^2_{K^*-(p+q)}$,
where p and q are from $\varepsilon_t \sim ARMA(p,q)$. ~~Since $a_t \overset{iid}{\sim} N(0,1)$, let $p=q=0$~~ $\varepsilon_t = (1-\phi B)a_t$,
so $p=0$, $q=1$.

<u>Note</u> If X_t and Y_t are WN, then $\sqrt{\underset{\substack{\uparrow \\ n = \text{\# of pairs of obs.}}}{n}}\,\hat{\rho}_{xy}(0) \sim N(0,1)$.
However, if X_t and Y_t exhibit autocorrelation, the sample cross-correlation function at lag k becomes
$\sqrt{n}\,\hat{\rho}_{xy}(k) \overset{\cdot}{\sim} N\left(0, 1+2\sum_{j=1}^{\infty}\rho_x(j)\rho_y(j)\right)$.

(Simpler case: $\hat{\rho}_x(k) \overset{\cdot}{\sim} N(0, \frac{1}{N})$, $k \neq 0$).